RELIABILITY ANALYSIS OF ON-LINE COMPUTER SYSTEMS WITH A SPARE PROCESSOR USING COMPUTER-AIDED ALGEBRAIC MANIPULATION

BY

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Abstract

This paper discusses the reliability of operation of an on-line computer system with a spare processor, described by a semi-Markov process model. Analytical solutions are obtained by using computer-aided algebraic manipulation techniques. The main purpose of the paper is to demonstrate that the difficulties of obtaining analytic solutions to Markov processes by standard techniques can be considerably reduced by the application of algebraic symbol manipulation languages. To the author's knowledge, the results of the reliability analysis are also new.

Results in this paper were obtained by using MACSYMA, available at MIT Mathematics Laboratory, supported by the Advanced Research Projects Agency (ARPA), Department of Defense, under Office of Naval Research Contract N00014-70-A-0362-0001. MACSYMA was accessed via the ARPA computer-communication network.
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I. INTRODUCTION

Consider an on-line computer system, such as one used to control newspaper production. To increase reliability of operation, the main processor is usually backed up by an identical unit. Without such a spare processor, a failure of the main processor can cause missing pulbication deadlines, resulting in revenue loss from advertisements. The reliability of operation can be enhanced by periodic maintenance of the processors. We assume that the processors fail only when in operation. When maintenance work is started on the active processor, the spare processor is put into operation. If the active processor does not fail till the end of the maintenance work, then the processor being maintained is set aside as the spare unit. If the active processor fails, then the processor being maintained or the spare unit is immediately put into operation. If a processor fails, then repair work is started on it immediately. If both processors fail, then the first to fail is repaired first.

This system can be characterized by four states listed in Table 1. Transitions between pairs of states occur at randomly distributed instants of time. Therefore, it is possible to describe the system by a semi-Markov process. The allowable transitions between pairs of states are shown in Fig. 1.

II. SEMI-MARKOV PROCESS MODEL

Let t_i denote the time spent by the process in state i before a transition to some other state occurs. We define the waiting-time distribution in state i as

$$W_i(t) = P[t_i \le t]$$

and the corresponding density function and mean by $w_i(t)$ and $\overline{w}_i$, respectively. Let the failure-time, repair-time and maintenance-time distribution be [2,3]

$$P[\text{Failure-time} \leq t] = 1 - \exp(-Lt) \ ,$$

$$P[\text{Repair-time} \leq t] = 1 - \exp(-Gt) \ ,$$

and

$$P[\text{Maintenance-time} \leq t] = 1 - \exp(-Ht) \ ,$$

respectively for $t \geq 0$ and zero otherwise. The maintenance schedule is assumed to be periodic with period T_0 and the following distribution

$$P[\text{Start of Maintenance} \leq t] = u(t - T_0) \ ,$$

where u is the unit-step function. The waiting-time distribution for each state can now be computed in a straightforward manner. As an example let us compute $W_1(t)$. Letting F denote the failure-time, we have

$$\begin{aligned} P[t_1 > t] &= P[\min(T_0,F) > t] \ , \\ &= P[T_0 > t]P[F > t] \ , \\ &= [1 - u(t-T_0)]\exp(-Lt) \ . \\ W_1(t) &= 1 - P[t_1 > t] \ , \\ &= 1 - [1 - u(t-T_0)]\exp(-Lt) \ , \end{aligned}$$

Hence, $w_1(t) = \dfrac{dW_1(t)}{dt} = [1 - u(t-T_0)]L\exp(-Lt) + \delta(t-T_0)\exp(-Lt)$

and $\overline{w}_1 = [1-\exp(-LT_0)]/L$. Figure 2 shows $W_1(t)$ as a function of t. The waiting-time distributions, their density functions and means are listed in Table 2. Let $p_{ij}(t)$ denote the conditional probability density function of a transition to state j in $[t,t+\Delta]$ given that the process entered state i at time zero and the next transition from state i occurs in $[t,t+\Delta]$ for sufficiently small $\Delta>0$. Then the probability density function of a transition from state i to state j

Table 1

STATE DESCRIPTION TABLE

State	Label
One active processor with a spare	1
Both processors down	2
One active processor with the other in maintenance	3
One active processor with no spare	4

Table 2

WAITING-TIME DISTRIBUTIONS

State	Distribution	Density	Mean
1	$1-[1-u(t-T_0)]\exp(-Lt)$	$[1-u(t-T_0)]L\exp(-Lt)$ $+\delta(t-T_0)\exp(-Lt)$	$[1-\exp(LT_0)]/L$
2	$1-\exp(-Gt)$	$G\exp(-Gt)$	$1/G$
3	$1-\exp(-(L+H)t)$	$(L+H)\exp(-(L+H)t)$	$1/(L+H)$
4	$1-\exp(-(L+G)t)$	$(L+G)\exp(-(L+G)t)$	$1/(L+G)$

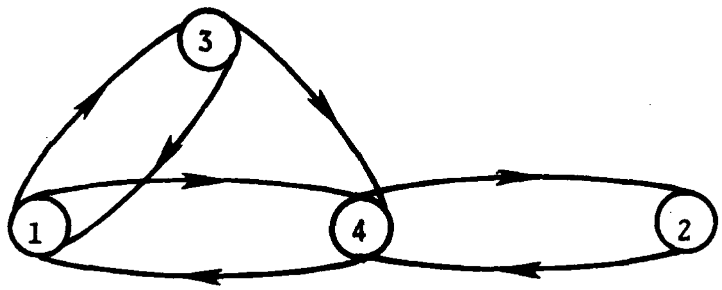

Figure 1

STATE TRANSITION DIAGRAM

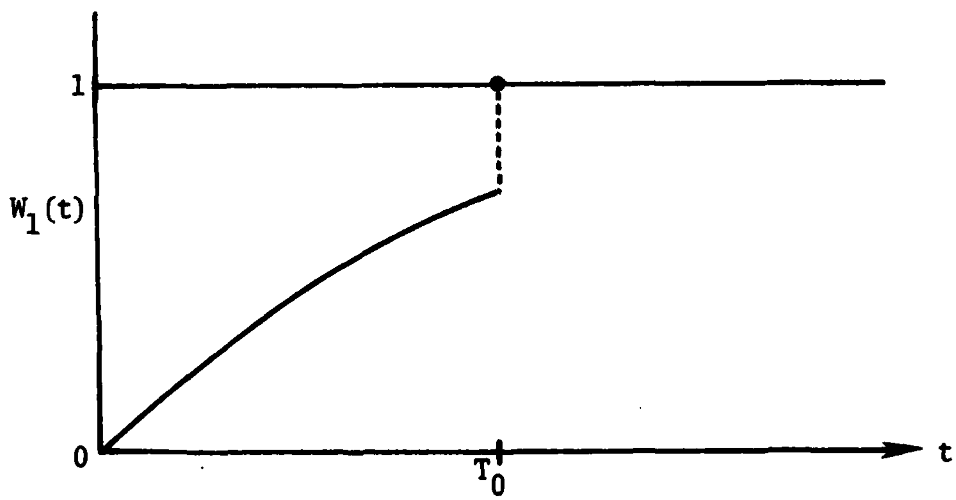

Figure 2

WAITING-TIME DISTRIBUTION FOR STATE 1

after waiting t units of time in state i is given by

$$c_{ij}(t) = p_{ij}(t)w_i(t).$$

The core matrix of a semi-Markov process is defined as $C(t)=[c_{ij}(t)]$ and it provides a complete probabilistic description of the process [1]. The core matrix of this process is shown in Table 3.

Let $e_{ij}(t)\Delta$ denote the probability that the process will enter state j in $[t,t+\Delta]$ given that it entered state i at time zero and let the entry matrix be $E(t)=[e_{ij}(t)]$. Then

$$E(s) = [I - C(s)]^{-1} ,$$

where I is the identity matrix and E(s) and C(s) are Laplace transforms of E(t) and C(t) respectively (see [1]). The matrix C(s) is shown in Table 4. Define

$$E = \lim_{s\to 0}[sE(s)] .$$

For a monodesmic process, such as the one discussed here, the rows of E are identical [1]. Let e_j denote the j^{th} element of any row of E. Then the limiting interval transition probability for state j, denoted by h_j, is given by

$$h_j = e_j\overline{w}_j .$$

Suppose the process has been operating unobserved for a long period of time. Then h_j is the probability of the event that the process will be in state j when observed next. The state occupancy statistics can also be obtained from E(s). The details of this and mean first-passage time computations can be found in [1].

The next section shows how analytic expressions can be obtained for h_j, state occupancy statistics, and mean first-passage times by using computer-aided

Table 3

CORE MATRIX

	1	2	3	4
1	0	0	$\delta(t-T_0)\exp(-Lt)$	$[u(t)-u(t-T_0)]L\exp(-Lt)$
2	0	0	0	$G\exp(-Gt)$
3	$H\exp(-(L+H)t)$	0	0	$L\exp(-(L+H)t)$
4	$G\exp(-(L+G)t)$	$L\exp(-(L+G)t)$	0	0

Table 4

LAPLACE TRANSFORM OF CORE MATRIX

	1	2	3	4
1	0	0	$\exp(-(S+L)T_0)$	$L(1-\exp(-(S+L)T_0)/(S+L)$
2	0	0	0	$G/(S+G)$
3	$H/(S+L+H)$	0	0	$L/(S+L+H)$
4	$G/(S+L+G)$	$L/(S+L+G)$	0	0

algebraic manipulation techniques. This material was generated interactively, by using the symbol manipulation language called MACSYMA available at the MIT mathematics laboratory, accessed via the ARPA computer communications network.

III. ANALYTIC SOLUTIONS VIA MACSYMA

The original computer outputs did not have any comments. The comments imbedded between the pairs of characters "/*" and "*/" are added to explain the procedure.

```
/* TIME:TRUE PRINTS CPU TIME USED IN EACH STEP IN MILLISECONDS:    */

(C1)                         TIME:TRUE $
TIME= 8 MSEC.
/* WAIT=ROW VECTOR OF MEAN WAITING TIMES:                        */
(C2)
                    WAIT:MATRIX([(1-%E^(-L*TO))/1,1/G,1/(L+H),1/(L+G)]);
TIME= 115 MSEC.
                                  [        - L TO                   ]
(D2)                              [ 1 - %E          1    1      1   ]
                                  [ ------------    -  -----  ----- ]
                                  [       L         G  L + H  L + G ]

/* SCORE=LAPLACE TRANSFORM OF THE CORE MATRIX = C(S).           */
/* SCORE IS ENTERED ROW BY ROW, EACH ROW ENCLOSED IN [ ].       */
/* %E DENOTES THE EXPONENTIAL "e" and ^ DENOTES EXPONENTIATION: */

(C3)
                    SCORE:MATRIX

        ([0,0,%E^(-))(S+L)*TO),(L/(S+1))*(1-%E^(-(S+L)*TO))],
         [0,0,0,G/(S+G)],
         [H/(S+L+H),0,0,L/(S+L+H)],
         [G/(S+L+G),//L/(S+L+G),0,0]);
TIME= 244 MSEC.
```

```
         [                                              - (S + L) TO ]
         [                       - (S + L) TO  L (1 - %E            ) ]
         [    0          0     %E              ---------------------- ]
         [                                             S + L          ]
         [                                                            ]
         [                                               G            ]
         [    0          0           0                 -----          ]
(D3)     [                                             S + G          ]
         [                                                            ]
         [    H                                          L            ]
         [ ---------     0           0               ---------        ]
         [ S + 1 + H                                 S + L + H        ]
         [                                                            ]
         [    G          L                                            ]
         [ ---------  ---------      0                   0            ]
         [ S + L + G  S + L + G                                       ]
```

/ SENTRY=E(S)=INVERSE OF (IDENTITY-SCORE). ^^ DENOTES NONCOMMUTATIVE */*

/ EXPONENTIATION. INVERSE OF MATRIX=MATRIX ^^-1: */*

```
(C4)
              SENTRY:(IDENT(4)-SCORE)^^-1 $

TIME= 56426 MSEC.
/* TEMPO=S*FIRST ROW OF SENTRY.RATSIMP IS AN OPERATOR USED FOR SIMPLIFICATION: */
(C5)          TEMPO:RATSIMP(S*ROW(SENTRY,1)) $

TIME= 128761 MSEC.
/* LMT=LIMIT OF TEMPO WHEN S-->0:          */
(C6)          LMT:LIMIT(TEMPO,S,0) $

LIMIT FASL DSK MACSYM BEING LOADED
LOADING DONE
TIME= 2251 MSEC.

/* LMTDST=ROW VECTOR OF STEADY STATE PROBABILITY DISTRIBUTION:      */


(C7)          LMTDST:RATSIMP(LMT*WAIT) $

TIME= 31529 MSEC.

(C8)          LMTDST[1,1];

TIME= 6 MSEC.
```

(D8)

$$\frac{(G^{2} L + G^{2} H)\ \%E^{L\ TO} - G^{2} L - G^{2} H}{(L^{3} + (H + G)\ L^{2} + (G\ H + G^{2})\ L + G^{2} H)\ \%E^{1\ TO} - H\ L^{2} - G\ H\ L - G^{2} H}$$

(C9)

```
LMTDST[1,2];
```

TIME= 5 MSEC.

(D9)

$$\frac{L^{2}}{L^{2} + G\ L + G^{2}}$$

(C10)

```
LMTDST[1,3];
```

TIME= 5 MSEC.

(D10)

$$\frac{G^{2} L}{(L^{3} + (H + G)\ L^{2} + (G\ H + G^{2})\ L + G^{2} H)\ \%E^{L\ TO} - H\ L^{2} - G\ H\ L - G^{2} H}$$

(C11)

```
LMTDST[1,4];
```

TIME= 5 MSEC.

(D11)

$$\frac{G\ L}{L^{2} + G\ L + G^{2}}$$

/ CHECK ON THE SUM OF THE PROBABILITIES OF LMTDST: */*

(C12)

```
RATSIMP(LMTDST.TRANSPOSE([1,1,1,1]));
```

TIME= 6407 MSEC.

```
(D12)                              1

/* TOCUP[1,2]=MEAN NO. OF TIMES STATE 2 IS VISITED IN [0,T] STARTING IN  */

/* STATE 1 AT TIME ZERO.  SOCUP[1,2]=LAPLACE TRANSFORM OF TOCUP[1,2]:    */


(C19)
              SOCUP[1,2]:RATSIMP(SENTRY[1,2]/S) $

TIME= 366 MSEC.
/* ILT OPERATOR COMPUTES THE INVERSE LAPLACE TRANSFORM:          */
(C20)         TOCUP[1,2]:ILT(SOCUP[1,2],S,T) $

LAPLAC FASL DSK MACSYM BEING LOADED
LOADING DONE
/* INFORMATION REQUESTED BY THE LAPLACE TRANSFORM ROUTINE:       */
IS   G L   POSITIVE, NEGATIVE OR ZERO?
/* ANSWER ENTERED FROM THE TERMINAL:                             */
POSITIVE;
TIME= 10625 MSEC.

/* COMPUTE TOCUP[1,2] FOR L=1, G=10, TO=1/10 AND H=10:           */

(C21)       %,L=1,G=10,TO=1/10,H=10;

TIME= 910 MSEC.
        - 11 T  89 SINH(SQRT(10) T)    109 COSH(SQRT(10) T)    10 T
(D21) %E       (------------------- + ---------------------) + ----
                 12321 SQRT(10)              12321             111

                                                                  109
                                                                - -----
                                                                  12321

/* TOCUP[1,2] HAS A LINEAR TERM IN T WHICH WILL BE DOMINANT FOR LARGE  */
/* VALUES OF T.  PART FUNCTION IS USED TO SELECT ANY PART OF AN        */
/* EXPRESSION.  THE LINEAR TERM IN T IS:                               */

(C22)
            PART(TOCUP[1,2],2);

TIME= 76 MSEC.

                                  2
                               G L  T
(D22)                        -------------
                              2          2
                             L  + G L + G
```

IV. DISCUSSION OF THE RESULTS

Two important parameters for estimating the reliability of operation of this system are h_1 and h_2, respectively the probabilities of being operational with a spare unit and completely shut-down, in the steady state. Let $D=(L+H)(L^2+GL+G^2)$. Then

$$h_1 = G^2[(L+H)(\exp(LT_0)-1)]/[D\exp(LT_0)-H(L^2+GL+G^2)] , \tag{1}$$

$$h_2 = L^2/(L^2+GL+G^2). \tag{2}$$

Also

$$h_3 = G^2L/[D\exp(LT_0)-H(L^2+GL+G^2)] , \tag{3}$$

and

$$h_4 = GL/(L^2+GL+G^2) . \tag{4}$$

During the interactive session, we verified that $h_1+h_2+h_3+h_4=1$ as it should be. As a further check on our results we consider the system without any preventive maintenance work. By letting T_0 go to infinity we eliminate all maintenance work in the future except that starting at time zero. The corresponding state transition diagram is shown in Fig. 3. In this case we have

$$h_1 = G^2(L+H)/D ,$$

$$h_2 = L^2/(L^2+GL+G^2) ,$$

$$h_3 = 0 ,$$

and

$$h_4 = GL/(L^2+GL+G^2) .$$

Now letting H go to infinity we eliminate state 3 from our process and obtain

$$h_1 = G^2/(L^2+GL+G^2)\,, \qquad (5)$$

$$h_2 = L^2/(L^2+GL+G^2)\,, \qquad (6)$$

$$h_3 = 0\,, \qquad (7)$$

and

$$h_4 = GL/(L^2+GL+G^2)\,. \qquad (8)$$

The steady state distribution for the resulting three state process can be computed by hand and they agree with equations (5), (6) and (8).

Now let us consider the effect of maintenance on h_1. We assume that when maintenance is done H is a constant and the reciprocal of the average failure-time L is a function of T_0, i.e., $L=F(T_0)$. F is assumed to be a nondecreasing function with $F(0)=L_0\neq 0$ and $F(\infty)=L_1<\infty$. Without maintenance, h_1 is computed from Equation (5) to be

$$\bar{h}_1 = G^2/(L_1^2+GL_1+G^2)\,. \qquad (9)$$

For a fair comparison, $\bar{h}_1$ should be compared with h_1+h_3 when maintenance is present. Using equations (1) and (3) we have

$$h_1^* + h_3^* = G^2/(F(T_0)^2+GF(T_0)+G^2)\;. \qquad (10)$$

Since F is assumed to be a nondecreasing function, $F(T_0) \leq F(\infty) = L_1$ and hence from equation (9) and equation (10) we have

$$\bar{h}_1 \leq h_1^* + h_3^* ,$$

i.e., maintenance increases the probability of the system being operational in the steady state.

Next let us consider a state-occupancy statistic of relevance to the reliability of the system. Let $N_{12}(T)$ denote the average number of times the process visits state 2 in the time interval [0,T], starting in state 1 at time zero. Then for large values of T

$$N_{12}(T) = GL^2T/(L^2+GL+G^2) . \tag{11}$$

Let
$$Q(L) = GL^2/(L^2+GL+G^2) = Gh_2 .$$

Then it is easy to verify that $Q(0)=0$, $Q(\infty)=G$, $dQ/dL>0$ for all $G,L>0$ and

$$\frac{d^2Q}{dL^2} > 0 \text{ for } G^3 > L^2(L+3G)$$

and

$$\frac{d^2Q}{dL^2} < 0 \text{ for } G^3 < L^2(L+3G)$$

We can sketch Q(L) as a function of L which is shown in Fig. 4. To minimize $N_{12}(T)$ we have to minimize Q(L) and this can be done by reducing T_0.

Next let us consider the average first-passage times between pairs of states. Let T_{ij} denote the average first-passage time from state i to state j. Then we define T_{21} as the recovery-time of the system from complete breakdown in state 2 to fully operational in state 1. The unknown quantities T_{ij} satisfy a set of linear algebraic equations [1] which can be easily solved by MACSYMA.

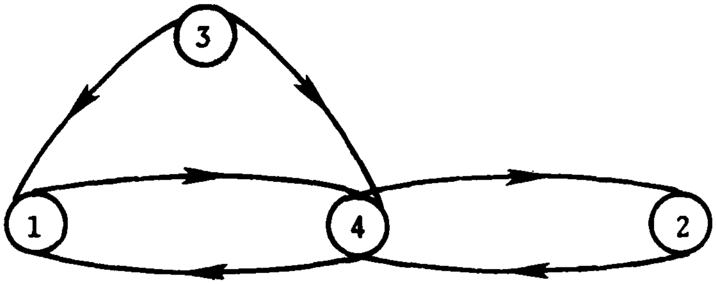

Figure 3

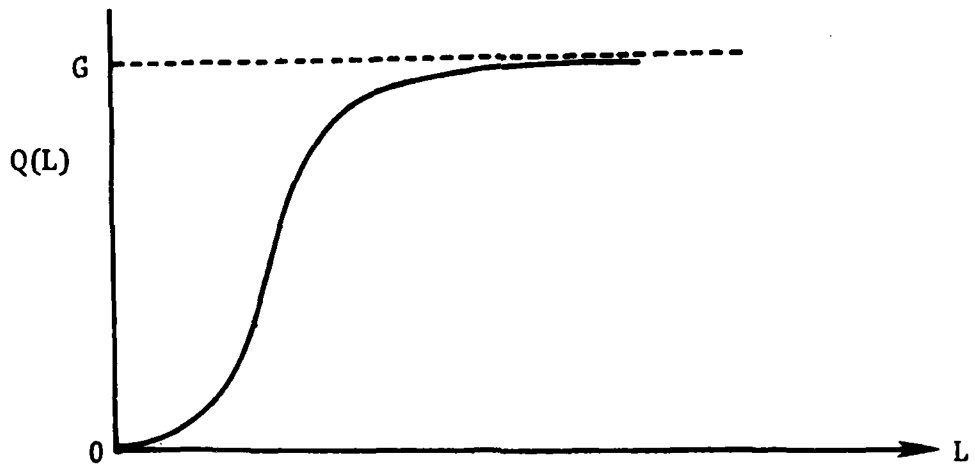

Figure 4

$$T_{21}(L) = (2/G) + (L/G^2) \geq 2/G \tag{12}$$

T_{21} is linearly related to L and can be reduced by decreasing L. From equations (11) and (12) we conclude that reducing T_0 will reduce $N_{12}(T)$ and T_{21} by decreasing L.

V. CONCLUSIONS

The same remarks made in [4] are also valid here.

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